Control Engineering 2016-2017
Exam 8 November 2016
Prof. C. De Persis

- You have **3 hours** to complete the exam.
- You **can** use books and notes but **not** smart phones, computers, tablets and the like.
- There are questions labeled as **Bonus**. These questions are optional and give you **extra** points if answered correctly.
- Please write down your Surname, Name, Student ID on each sheet.
- You will be given 2 sheets. If you need more, please ask. Please hand in **all the sheets** that you have used and the **text of the exam**.
- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page.

For the grader only

	Exercise 1	Exercise 2	Exercise 3	Exercise 4	Exercise 5	Total
Points	/12	/16	/16	/10	/16	/70
Bonus	⋆⋆⋆	⋆⋆⋆	/5	⋆⋆⋆	/5	/10

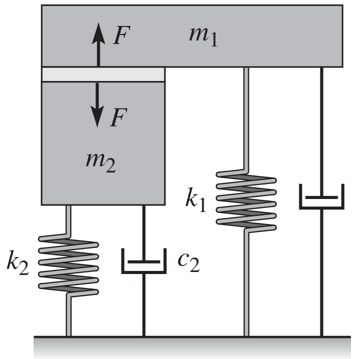

Figure 1: From Aström-Murray, Second Edition (2016), Chapter 4.

1. [12pts] The goal of the problem is to derive the equations of motion of an atomic force microscope (AFM) with piezotube depicted in Figure 1 using the Euler-Lagrange equations of motion without considering the gravity forces. The AFM is modeled as two masses separated by an ideal piezo element that exerts a force F on both masses as illustrated in the figure. The generalized coordinate of the system is $q = (z_1, z_2)$, where z_i, $i = 1, 2$, is the vertical position of the center of mass of mass i, whereas the generalized velocity is $\dot{q} = (\dot{z}_1, \dot{z}_2)$.
To obtain these equations of motion, answer the following questions:

   (a) [1pt] Determine the total kinetic co-energy $T_m^*(q, \dot{q})$ of the system.

   (b) [1pt] Determine the potential function $V(q)$.

   (c) [1pt] Determine the Lagrangian function $L(q, \dot{q})$.

   (d) [1pt] Determine the Rayleigh dissipation function.

   (e) [2pts] Determine the vector τ of external generalized forces.

   (f) [3pts] Determine the Euler-Lagrange equations of motion.

   (g) [3pts] Define $u = F$ as the input and $y = q_1$ the output of the system. Choose the state variable vector x and express the system as the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

   giving the explicit values of the matrices A, B, C, D.

Note that $q_1 = z_1, q_2 = z_2$ in the following.

(a) [1pt] The total kinetic co-energy $T_m^*(q, \dot{q})$ is given by $\frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 = \frac{1}{2}\dot{q}^T M\dot{q}$, with $M = \text{diag}(m_1, m_2)$.

(b) [1pt] $V(q) = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 q_2^2$.

(c) [1pt] $L(q, \dot{q}) = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - \frac{1}{2}k_1 q_1^2 - \frac{1}{2}k_2 q_2^2$.

(d) [1pt] $D(\dot{q}) = \frac{1}{2}c_1\dot{q}_1^2 + \frac{1}{2}c_2\dot{q}_2^2$.

(e) [2pts] $\tau = \begin{bmatrix} F \\ -F \end{bmatrix}$.

(f)

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} m_1\dot{q}_1 \\ m_2\dot{q}_2 \end{bmatrix} \quad [0.5\text{pt}]$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} m_1\ddot{q}_1 \\ m_2\ddot{q}_2 \end{bmatrix} \quad [0.5\text{pt}]$$

Also

$$\frac{\partial L}{\partial q} = -\begin{bmatrix} k_1 q_1 \\ k_2 q_2 \end{bmatrix} \quad [0.5\text{pt}]$$

and overall

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \begin{bmatrix} m_1\ddot{q}_1 \\ m_2\ddot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 q_1 \\ k_2 q_2 \end{bmatrix} = -\begin{bmatrix} c_1\dot{q}_1 \\ c_2\dot{q}_2 \end{bmatrix} + \begin{bmatrix} F \\ -F \end{bmatrix} \quad [1\text{pt}]$$

having born in mind that

$$\frac{\partial D}{\partial \dot{q}} = \begin{bmatrix} c_1\dot{q}_1 \\ c_2\dot{q}_2 \end{bmatrix} \quad [0.5\text{pt}]$$

(g) The EL equations of motion are

$$\begin{aligned} m_1\ddot{q}_1 + c_1\dot{q}_1 + k_1 q_1 &= u \\ m_2\ddot{q}_2 + c_2\dot{q}_2 + k_2 q_2 &= -u \end{aligned} \quad [0.5\text{pt}]$$

The choice of state variables

$$x_1 = q_1, \quad x_2 = \dot{q}_1, \quad x_3 = q_2, \quad x_4 = \dot{q}_2 \quad [0.5\text{pt}]$$

returns the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{k_1}{m_1}x_1 - \tfrac{c_1}{m_1}x_2 + \tfrac{1}{m_1}u \\ \dot{x}_3 &= x_3 \\ \dot{x}_4 &= -\tfrac{k_2}{m_2}x_3 - \tfrac{c_2}{m_2}x_4 + \tfrac{1}{m_2}u \end{aligned} \quad [1\text{pt}]$$

i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ -\frac{1}{m_2} \end{bmatrix}, \quad [1\text{pt}]$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

2. [16pts] (**Alcohol metabolism**) The metabolism of alcohol in the body can be modeled by the normalized nonlinear compartmental model

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + u \\ \dot{x}_2 &= b(x_1 - x_2) - \frac{cx_2}{d + x_2} + u \end{aligned} \tag{1}$$

where:

- $x_1, x_2 \in \mathbb{R}$ are the concentrations of alcohol in the compartments;
- $u \in \mathbb{R}$ is the intravenous and gastrointestinal injection rate;
- a, b, c, d are all positive and constant parameters.

Answer the following questions:

(a) [4.5pts] Given a constant input $\bar{u}$, with $\bar{u}$ a positive constant, determine the equilibrium $\bar{x} = (\bar{x}_1 \ \bar{x}_2)^T$ of the system. State a condition on $\bar{u}$ that guarantees the equilibrium vector $\bar{x}$ to have both positive components.

(b) [3.5pts] Linearize the dynamics of the compartmental model around the equilibrium pair $(\bar{x}, \bar{u})$ obtained using the identities

$$b\frac{\bar{u}}{a} + \bar{u} = \frac{c}{2}, \quad \bar{u} = ad, \quad c = 4bd,$$

that is determine the matrices A, B in

$$\dot{\delta x} = A\delta x + B\delta u.$$

(c) [3pts] For the linearized system obtained in Question (b)

$$\dot{\delta x} = A\delta x + B\delta u,$$

set $\delta u = 0$ (this corresponds to set $u = \bar{u}$). Determine whether the origin of the resulting system is asymptotically stable, stable or unstable. What is the expected behavior of the solutions of the original nonlinear system (1) with $u = \bar{u}$ that start sufficiently close to the equilibrium $\bar{x}$? Motivate your answer.

(d) [5pts] Set $a = 3$, $b = 2$ and remember that δu is a scalar (same intravenous and gastrointestinal injection rate). Let the output be

$$\delta y = \delta x_1.$$

Compute the unitary step response of the linearized system, that is the output response of the linearized system when $\delta x(0) = 0$ and $\delta u(t) = 1$ for all $t \geq 0$. **Hint** If you did not determine the matrices A, B in Question (b), then use the following data

$$A = \begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Solve the equation (bars are omitted from the variables)

$$\begin{aligned} 0 &= a(x_2 - x_1) + u \\ 0 &= b(x_1 - x_2) - \frac{cx_2}{d + x_2} + u \end{aligned} \qquad [0.5pt]$$

From the first equation

$$x_1 - x_2 = \frac{u}{a} \qquad [0.5pt]$$

which replaced in the second one gives

$$0 = b\frac{u}{a} - \frac{cx_2}{d + x_2} + u \qquad [1pt]$$

returning

$$x_2 = \frac{(\frac{b}{a} + 1)ud}{c - (\frac{b}{a} + 1)u}. \qquad [1pt]$$

Hence

$$x_1 = \frac{(\frac{b}{a} + 1)ud}{c - (\frac{b}{a} + 1)u} + \frac{u}{a} \qquad [0.5pt]$$

x_2 (and therefore x_1) is positive provided that

$$(\frac{b}{a} + 1)u < c. \qquad [1pt]$$

(b) The Jacobians of the right-hand side of (1) are

$$[1pt]\ \frac{\partial f}{\partial x} = \begin{bmatrix} -a & a \\ b & -b - \dfrac{cd}{(d + x_2)^2} \end{bmatrix}, \qquad [0.5pt]\ \frac{\partial f}{\partial u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Under the condition

$$(\frac{b}{a} + 1)u = \frac{c}{2}, \quad u = ad,$$

the equilibrium is

$$[1pt]\ x_2 = d, \quad x_1 = 2d.$$

Hence, when the Jacobians are evaluated at this particular equilibrium, they return

$$[1pt]\ A = \begin{bmatrix} -a & a \\ b & -b - \dfrac{c}{4d} \end{bmatrix} = \begin{bmatrix} -a & a \\ b & -2b \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(c) The eigenvalues of A are given by the roots of

$$s^2 + (a + 2b)s + ab = 0 \qquad [0.5\text{pt}]$$

namely

$$s_{1,2} = \frac{-(a+2b) \pm \sqrt{(a+2b)^2 - 4ab}}{2} = \frac{-(a+2b) \pm \sqrt{a^2 + 4b^2}}{2} \qquad [0.5\text{pt}]$$

[1.5pt] Since a, b are positive parameters, the eigenvalues are real distinct and strictly negative. Hence, the origin of the linearized system is asymptotically stable. [0.5pt] The solutions that start sufficiently close to the origin converge asymptotically to it.

(d) For the given values of a, b, the linearized system matrices are

$$A = \begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad [1\text{pt}]$$

Output response to be computed similarly as in Tutorial 7, Exercise 5.

Possibility 1: (Matrix exponential) The matrix A can be decomposed as

$$A = VDV^{-1}$$

with

$$V = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}. \qquad [1\text{pt}]$$

Then note that the output-response is given by

$$\begin{aligned} y(t) &= Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ &= CV\int_0^t e^{D(t-\tau)}V^{-1}Bd\tau \\ &= CV\int_0^t e^{D(t-\tau)}d\tau V^{-1}B \\ &= CV\left|-D^{-1}e^{D(t-\tau)}\right|_{\tau=0}^{\tau=t} V^{-1}B \\ &= -CVD^{-1}V^{-1}B + CVD^{-1}e^{Dt}V^{-1}B \qquad [1\text{pt}] \end{aligned}$$

Observe that

$$CVD^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{6} & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -3 \end{bmatrix}$$

$$V^{-1}B = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} \qquad [1\text{pt}]$$

Hence

$$
\begin{aligned}
y(t) &= -CVD^{-1}V^{-1}B + CVD^{-1}e^{Dt}V^{-1}B \\
&= -\begin{bmatrix}\frac{1}{6} & -3\end{bmatrix}\begin{bmatrix}\frac{1}{5}\\ \frac{2}{5}\end{bmatrix} + \begin{bmatrix}\frac{1}{6} & -3\end{bmatrix}\begin{bmatrix}e^{-6t} & 0\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}\frac{1}{5}\\ \frac{2}{5}\end{bmatrix} \\
&= \frac{1}{30}\left(e^{-6t} - 36e^{-t} + 35\right). \qquad \text{[1pt]}
\end{aligned}
$$

Possibility 2: (Transfer function and Inverse Laplace transform). The transfer function is given by

$$
\begin{aligned}
G(s) = C(sI-A)^{-1}B &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s+3 & -3\\ -2 & s+4\end{bmatrix}^{-1}\begin{bmatrix}1\\ 1\end{bmatrix} \\
&= \frac{1}{s^2+7s+6}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s+4 & 3\\ 2 & s+3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} \\
&= \frac{1}{(s+6)(s+1)}\begin{bmatrix}s+4 & 3\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{s+7}{(s+6)(s+1)} \qquad \text{[1.5pt]}
\end{aligned}
$$

The output transfer function is equal to

$$
Y(s) = G(s)\frac{1}{s} = \frac{s+7}{s(s+6)(s+1)} = \frac{a}{s} + \frac{b}{s+6} + \frac{c}{s+1} \qquad \text{[1pt]}
$$

for [0.5pt] $a = \frac{7}{6}, b = \frac{1}{30}, c = -\frac{6}{5}$ which gives again the time response

$$
y(t) = \frac{7}{6} + \frac{1}{30}e^{-6t} - \frac{6}{5}e^{-t} = \frac{1}{30}\left(e^{-6t} - 36e^{-t} + 35\right). \qquad \text{[1pt]}
$$

3. [16pts] Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= Cx &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} x, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^2$ is the state vector, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the measured output.

(a) [1pt] Determine the reachability matrix W_r and discuss whether the system is reachable or not.

(b) [1pt] Determine the reachable canonical form of the state space equation.

(c) [1pt] Determine the reachability matrix $\tilde{W}_r$ of the reachable canonical form.

(d) [1pt] Determine the gain matrix K such that the eigenvalues of $A - BK$ are equal to the values $\{-1, -1\}$. Write explicitly the feedback $u = -Kx$.

(e) [2pts] Determine the observability matrix W_o of system (2). Assume that you can choose either a sensor that measures x_1 or a sensor that measures x_2, but not both of them simultaneously. Which sensor would you choose to guarantee the observability of the system? Give the corresponding output matrix C. Motivate your answer.

(f) [1pt] Write the observable canonical form and compute the observability matrix $\tilde{W}_o$ of the system in the observable canonical form.

(g) [2pts] Determine the observer gain L that makes the characteristic polynomial of the matrix $A - LC$ coincide with the polynomial $(s+3)^2$ and provide the explicit expression of the observer, namely provide the matrices F, G, H in the dynamical system

$$\dot{\hat{x}} = F\hat{x} + Gy + Hu.$$

(h) [4pts] Consider now the case in which $y = x_1$. We want to design a new asymptotic observer of dimension 1 instead of 2. Namely, consider the dynamical system (the so-called reduced order observer)

$$\begin{aligned} \dot{\xi} &= -g\xi + u - g(1+g)y \\ \hat{x}_2 &= \xi + gy \end{aligned} \tag{3}$$

where g is a constant parameter to design, $\xi \in \mathbb{R}$ is the state variable of the reduced order observer, $\hat{x}_2 \in \mathbb{R}$ is the estimate of the unmeasured state x_2. Introduce the estimation error $e = x_2 - \hat{x}_2$ and derive its dynamics $\dot{e} = f(e)$ explicitly stating the function $f(e)$. Then determine all the values of the parameter g that guarantee the estimation error e to converge exponentially to zero.

(i) [3pts] Using the reduced order observer derived in Question (h) and the state feedback controller $u = -Kx$ derived in Question (d), design a dynamic output feedback controller of dimension 1, namely a controller of the form

$$\begin{aligned} \dot{\xi} &= a\xi + by \\ u &= c\xi + dy \end{aligned} \tag{4}$$

with a, b, c, d parameters to be determined, that asymptotically stabilizes the closed-loop system.

Hint Start from the reduced order observer (3) and determine an expression of u that only uses the measurement y and the controller state ξ. Use the same idea as in the design of the output feedback controller [Textbook, Chapter 7].

(j) **(Bonus) [5pts]** Sketch the main steps that you would take to prove that the proposed dynamic controller yields an asymptotically stable closed-loop system.

Hint There is no need to have answered Question (h) to answer this one. If you did not answer Question (d), use as state feedback controller $u = -k_1 x_1 - k_2 x_2$, with k_1, k_2 two real numbers.

(a)

$$[0.5\text{pts}]\ W_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\ ;\ [0.5\text{pts}]\ \det\ W_r = -1 \neq 0 \rightarrow \text{system is reachable}$$

(b)

$$\det(sI - A) = s^2 - s =: s^2 + a_1 s + a_2 \qquad [0.5\text{pts}]$$

hence

$$\dot{z} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\ , \qquad [0.5\text{pts}]$$

(c)

$$\tilde{W}_r = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \qquad [1\text{pt}]$$

(d) The desired characteristic polynomial is

$$[0.5\text{pt}]\ (s+1)^2 = s^2 + 2s + 1 =: s^2 + p_1 s + p_2$$

$$K = \begin{bmatrix} p_1 - a_1 & p_2 - a_2 \end{bmatrix} \tilde{W}_r W_r^{-1}$$

$$p_1 = 2, \qquad p_2 = 1$$

$$\tilde{W}_r = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$W_r^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence

$$\begin{aligned} K &= \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ [0.5\text{pt}] \qquad &= \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 \end{bmatrix} \end{aligned}$$

The feedback is

$$u = -4x_1 - 3x_2.$$

Check

$$\begin{aligned} A - BK &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix} \end{aligned}$$

whose characteristic polynomial is indeed $s^2 + 2s + 1$.

(e) $W_0 = \begin{bmatrix} c_1 & c_2 \\ c_1 & c_1 \end{bmatrix}$ [0.5 pts]; $\det W_0 = c_1(c_1 - c_2)$ [0.5 pts]. To be observable, it must be $c_1 \neq 0$ and $c_1 \neq c_2$. Since we can have either (i) $c_1 = 1$ and $c_2 = 0$ or (ii) $c_1 = 0$ and $c_2 = 1$, the only possible case is (i), that is

$$[1\text{pt}]\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(f)

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z \quad [0.5\text{ pts}] \qquad \tilde{W}_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \ [0.5\text{ pts}] \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z \end{aligned}$$

Given the previous choice of C, the system is already in the observer canonical form. Therefore $\tilde{W}_o = W_o$.

(g)

$$\left.\begin{aligned} &L = W_0^{-1}\tilde{W}_0 \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix} \\ &(s+3)^2 = s^2 + 6s + 9 = s^2 + p_1 s + p_2 \end{aligned}\right) [0.5\text{ pt}]$$

$$L = W_0^{-1}\tilde{W}_0 \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix} = \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix} = \begin{bmatrix} 6+1 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \quad [0.5\text{ pt}]$$

The observer takes the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \implies \\ [0.5\text{pt}] \quad &= (A - LC)\hat{x} + Bu + Ly \\ F = A - LC,& \quad G = L, \quad H = B \end{aligned}$$

Hence

$$\begin{aligned} F &= A - LC \\ [0.5\text{pts}] \quad &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 7 \\ 9 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 1 \\ -9 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 7 \\ 9 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

(h) We have

$$\begin{aligned} \dot{e} = \dot{x}_2 - \dot{\hat{x}}_2 &= u - \dot{\xi} - g\dot{y} = u - (-g\xi + u - g(1+g)y) - g\dot{x}_1 \\ &= u - (-g\xi + u - g(1+g)x_1) - g(x_1 + x_2) \\ &= g(\hat{x}_2 - gx_1) + g(1+g)x_1 - gx_1 - gx_2 \\ &= -g(x_2 - \hat{x}_2) - g^2 x_1 + g(1+g)x_1 - gx_1 = -ge \qquad [3\text{pts}] \end{aligned}$$

Hence, the estimation error exponentially converges to zero for all $g > 0$ [1pts].

(i) The proposed controller of dimension 1 would be

$$\begin{array}{rrcl} & \dot{\xi} & = & -g\xi + u - g(1+g)y \\ \text{[1pts]} & \hat{x}_2 & = & \xi + gy \\ & u & = & -4y - 3\hat{x}_2 \end{array}$$

and therefore

$$\begin{array}{rrcl} & \dot{\xi} & = & -g\xi - 4y - 3(\xi + gy) - g(1+g)y \\ \text{[1.5pts]} & & = & -(g+3)\xi - (4 + 4g + g^2)y \\ & u & = & -3\xi - (3g+4)y \end{array}$$

Hence, [0.5pts] $a = -(g+3), b = -(4+4g+g^2), c = -3, d = -(3g+4)$.

(j) **Bonus:** [5 pts]

i. First write down the closed-loop system in the variables x, ξ. Observe that

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u = c\xi + dx_1 \qquad \text{[1pt]} \\ \dot{\xi} &= a\xi + bx_1 \end{aligned} \tag{5}$$

ii. By defining $e = x_2 - \xi - gx_1$, the system (5) is transformed to

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u = c(x_2 - e - gx_1) + dx_1 \qquad \text{[1.5pts]} \\ \dot{e} &= -ge \end{aligned}$$

which in state-space form amounts to

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 0 \\ d - gc & c & -c \\ 0 & 0 & -g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -4 & -3 & 3 \\ 0 & 0 & -g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ -k_1 & -k_2 & k_2 \\ 0 & 0 & -g \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e \end{bmatrix} \qquad \text{[1.5pts]} \end{aligned}$$

iii. [1pt] By construction the matrix is Hurwitz, and this proves the claim.

Hint (It should probably be $u = -k_1x_1 - k_2\hat{x}_2$!) if we initialized with the hint we obtain

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u = -k_1x_1 - k_2\hat{x}_2 = -k_1x_1 - k_2x_2 + k_2e \qquad \text{[3pt]} \\ \dot{e} &= f(e). \end{aligned} \tag{6}$$

[1pt] With e converging exponentially to zero and [1pt] having the matrix

$$\begin{bmatrix} 1 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

to be Hurwitz, stability of the closed-loop system is guaranteed.

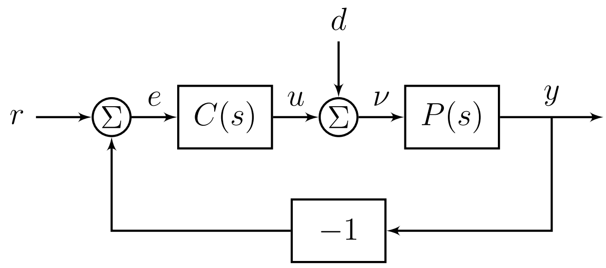

Figure 2: Negative feedback block diagram considered in Problem 4.

4. [10 pts] Consider the negative feedback control system in Figure 2.

Let $P(s)$ be the transfer function of the linearized model of a car studied e.g. in Lecture 13, namely

$$P(s) = \frac{b}{s+a}$$

where a, b are positive parameters. You are asked to design the cruise control of the car in a scenario in which the car is riding on a road whose slope changes periodically, producing a periodic disturbance $d(t)$ that takes the form

$$d(t) = \overline{d}\sin(\overline{\omega}t),$$

where $\overline{d}, \overline{\omega}$ are two positive parameters, of which only $\overline{\omega}$ is known. Consider initially a cruise control given by a PI control of the form

$$C(s) = \frac{k_p s + k_i}{s}.$$

Assume that the gains k_p, k_i have been designed in such a way that the closed-loop system is asymptotically stable.

(a) [4pts] Determine the transfer function $G_{ed}(s)$ from the disturbance d to the error e and the steady state error response to the disturbance $d(t) = \overline{d}\sin(\overline{\omega}t)$. **Hint** To determine the steady state error response, express $G_{ed}(i\omega)$ in the polar form $M(\omega)\mathrm{e}^{i\theta(\omega)}$.

(b) [2pts] Fix the values $a = 0.0101$, $b = 1.32$, $k_p = 0.5$, $k_i = 0.0051$. Give the numerical expression of the steady state error response derived before. Evaluate the gain[1] $M(\overline{\omega})$ at $\overline{\omega} = 0.1$ rad/sec.

(c) [2pts] We want now to design a new controller $C(s)$. Consider the factorization $d_{C1}(s)d_{C2}(s)$ of the controller denominator, and let $d_{C2}(s) = s^2 + \overline{\omega}^2$. Assume that $\frac{n_C(s)}{d_{C1}(s)}$ is the idealized PID control

$$\frac{n_C(s)}{d_{C1}(s)} = k_p + \frac{k_i}{s} + k_d s.$$

Determine, if possible, the gains of the PID control such that the denominator of the transfer function of the closed-loop system is $s^4 + s^3 + 3s^2 + 2s + 1$. To answer this question, set $a = b = \overline{\omega} = 1$.

[1] This is the attenuation/amplification factor of the disturbance magnitude.

(d) [2pts] Check that the steady state error response is zero when $r = 0$ and $d(t) = \overline{d}\sin(t)$.

(a)

$$\begin{aligned} G_{ed}(s) &= -\frac{P(s)}{1+C(s)P(s)} \\ &= -\frac{n_P(s)d_C(s)}{d_P(s)d_C(s)+n_P(s)n_C(s)} \end{aligned}$$

hence

$$\begin{aligned} G_{ed}(s) &= -\frac{\frac{b}{s+a}}{1+\frac{k_i+k_p s}{s}\frac{b}{s+a}} \\ &= -\frac{bs}{s(s+a)+b(k_i+k_p s)} \\ &= -\frac{bs}{s^2+(a+bk_p)s+bk_i} \end{aligned}$$

Since the closed-loop system is asymptotically stable the steady state error response is given by

$$e(t) = M(\overline{\omega})\sin(\overline{\omega}t + \theta(\overline{\omega})).$$

Compute

$$\begin{aligned} G_{ed}(i\omega) &= -\frac{ib\omega}{(-\omega^2+bk_i)+i(a+bk_p)\omega} \\ &= -\frac{ib\omega[(-\omega^2+bk_i)-i(a+bk_p)\omega]}{(-\omega^2+bk_i)^2+(a+bk_p)^2\omega^2} \\ &= -\frac{ib\omega(-\omega^2+bk_i)+b\omega(a+bk_p)\omega}{(-\omega^2+bk_i)^2+(a+bk_p)^2\omega^2} \\ &= \frac{ib\omega(\omega^2-bk_i)-b\omega(a+bk_p)\omega}{(-\omega^2+bk_i)^2+(a+bk_p)^2\omega^2} \end{aligned}$$

Hence

$$G_{ed}(i\omega) = \underbrace{\frac{b\omega}{\sqrt{(-\omega^2+bk_i)^2+(a+bk_p)^2\omega^2}}}_{M(\omega)} \mathrm{e}^{\,i\underbrace{\arctan\frac{(\omega^2-bk_i)}{-(a+bk_p)\omega}}_{\theta(\omega)}}$$

from which

$$e(t) = M(\overline{\omega})\sin(\overline{\omega}t + \theta(\overline{\omega}))\overline{d}.$$

(b)

$$
\begin{aligned}
M(\bar{\omega}) &= \frac{b\bar{\omega}}{\sqrt{(-\bar{\omega}^2 + bk_i)^2 + (a + bk_p)^2\bar{\omega}^2}} \\
&= \frac{1.32\bar{\omega}}{\sqrt{(-\bar{\omega}^2 + 1.32 \cdot 0.0051)^2 + (0.0101 + 1.32 \cdot 0.5)^2\bar{\omega}^2}} \\
&= \frac{1.32\bar{\omega}}{\sqrt{(-\bar{\omega}^2 + 0.0067)^2 + 0.45\bar{\omega}^2}} = 1.96752 \quad [1\text{pt}] \\
\theta(\bar{\omega}) &= \arctan \tfrac{(\bar{\omega}^2 - bk_i)}{-(a+bk_p)\bar{\omega}} = -0.0487302\,\text{rad} \quad [1\text{pt}]
\end{aligned}
$$

(c) The transfer function from d to y is given by

$$
\begin{aligned}
G_{yd} = \frac{P}{1+PC} = \frac{b}{s+a+bC} &= \frac{b(s^2+1)s}{(s^2+1)(s+a)s + b(k_p s + k_i + k_d s^2)} \\
&= \frac{(s^2+1)s}{(s^2+1)(s+1)s + k_p s + k_i + k_d s^2} \\
&= \frac{(s^2+1)s}{s^4 + s + s^3 + s^2 + k_p s + k_i + k_d s^2} \\
&= \frac{(s^2+1)s}{s^4 + s^3 + (k_d+1)s^2 + (k_p+1)s + k_i}
\end{aligned}
$$

Hence, $k_p = 1$, $k_i = 1$, $k_d = 2$.

(d) The Laplace transform of $d(t) = \overline{d}\sin(t)$ is given by

$$
D(s) = \frac{\overline{d}}{s^2+1}
$$

Hence

$$
\begin{aligned}
Y(s) = G_{yr}(s)D(s) &= \frac{(s^2+1)s}{s^4+s^3+3s^2+2s+1}\frac{\overline{d}}{s^2+1} \\
&= \frac{s\overline{d}}{s^4+s^3+3s^2+2s+1}
\end{aligned}
$$

Since closed-loop system is assumed to be stable, the Final Value Theorem can be used to obtain

$$
y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s^2\overline{d}}{s^4+s^3+3s^2+2s+1} = 0.
$$

5. [16pts] Consider a negative feedback system as in Figure 2, where $d = 0$, and the process transfer function is

$$P(s) = \frac{0.5s + 1}{s(s - 1)}.$$

Note the unstable pole due to the monomial $s - 1$ at the denominator.

(a) [3pts] Consider the Bode diagrams in Figure 3, 4, 5. Determine which Bode diagram (A, B, or C) corresponds to the given process transfer function $P(s)$. Motivate your answer.
Hint To give the right answer it is essential that you recall the phase plot corresponding to an unstable pole. This has been discussed in [Lecture 11, Slide 12]. An example of Nyquist plot of a transfer function with an unstable pole has been discussed in [Lecture 12, Slide 20].
Note Matlab, which was used to generate the Bode diagrams, use the convention that negative numbers have phase $-180°$ instead of $180°$.

(b) [2pts] Draw the Nyquist plot (including the circle at infinity and specifying the direction of the curve as the frequency ω goes from $-\infty$ to $+\infty$) corresponding to the Bode diagram that you have chosen in the previous question.

(c) [3pts] For the stability study to follow, it is convenient to compute analytically the phase crossover frequency ω_{pc}. To this purpose, compute the nonzero frequency ω_{pc} at which the imaginary part of $P(i\omega_{pc})$ is zero. Then compute the gain $|P(i\omega_{pc})|$.

(d) [2pts] Using the Nyquist plot and the gain $|P(i\omega_{pc})|$ computed before, explain whether or not the closed-loop system is stable. Motivate your answer.
Hint For the stability study, it is useful to know the gain $|P(i\omega_{pc})|$. If you have not found it, go the next question, answer it and use the finding there to finalise the stability study.

(e) [2pts] Compute the transfer function of the closed-loop system and discuss whether or not its poles have all strictly negative real part. Is your finding consistent with the answer to the previous question?

(f) [4pts] Design a P (proportional) controller

$$C(s) = k_p$$

such that the closed-loop system

i. is asymptotically stable; and has
ii. a zero steady state error response to a step input and
iii. a constant e_{steady} steady state error response to a unitary ramp input such that $|e_{steady}| \leq 0.1$.

(g) **(Bonus) [3pts]** Consider again the P controller $C(s) = k_p$ that stabilizes the system. Motivate using the Nyquist plot why the value of k_p you found in Question (f) i. makes the closed-loop system asymptotically stable.

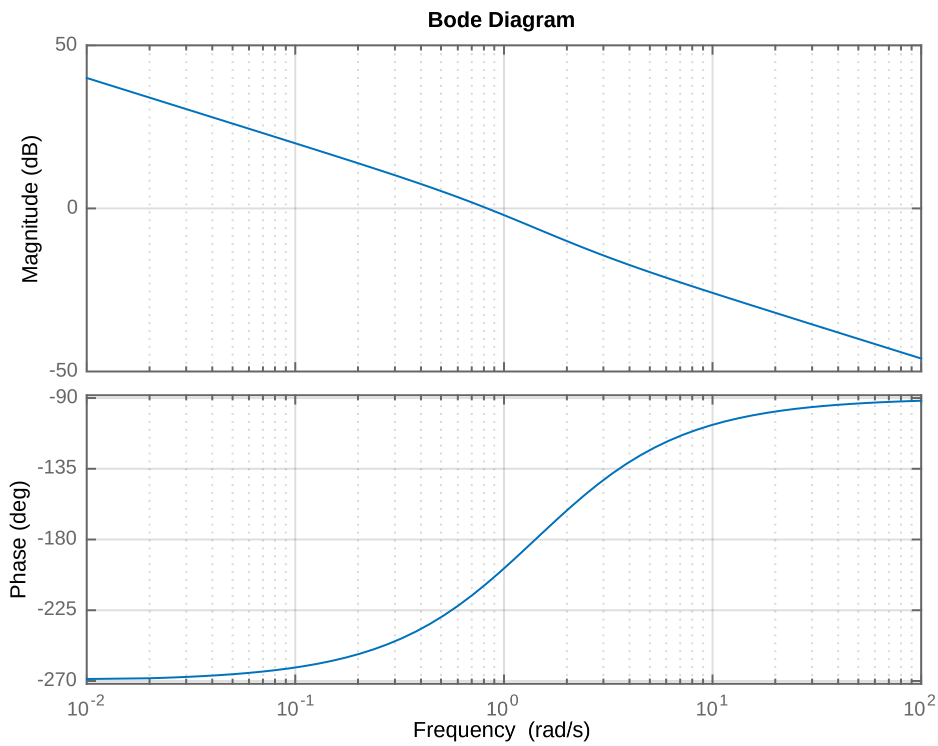

Figure 3: Bode diagram A.

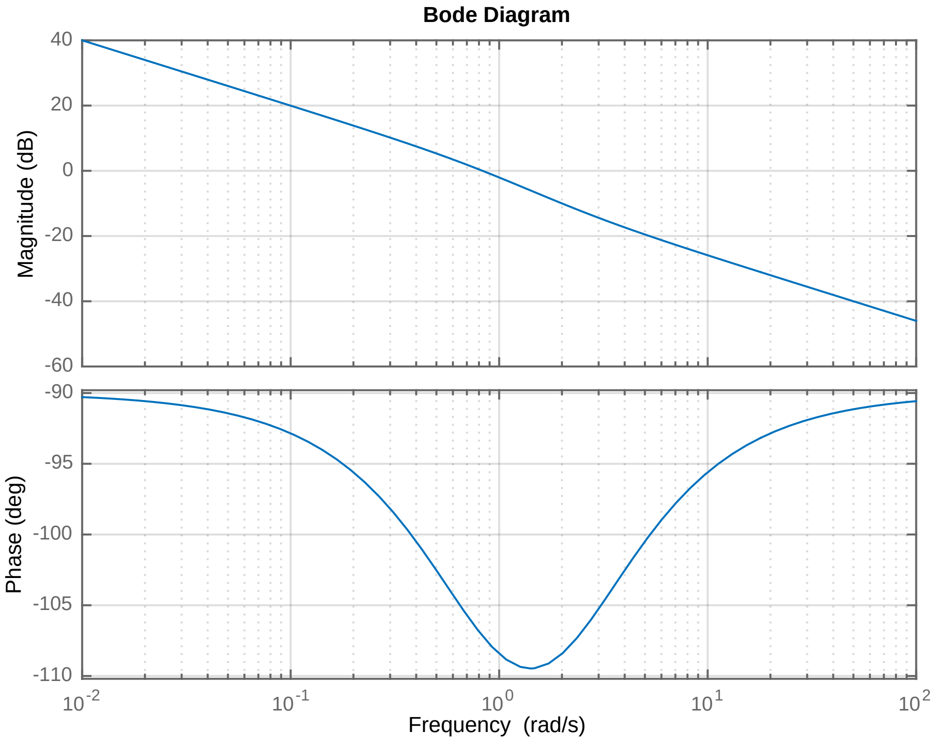

Figure 4: Bode diagram B.

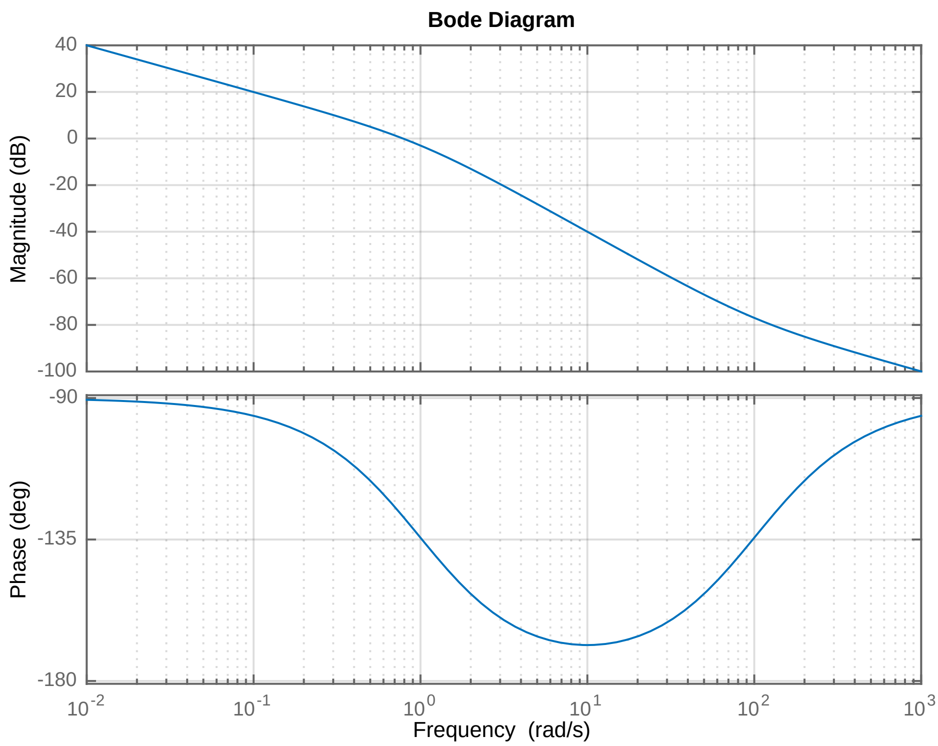

Figure 5: Bode diagram C.

(a) The correct one is the Bode diagram in Figure 3, as it is inferred by the fact that the unstable pole $s+1$ contribute a positive phase, rather than a negative one, thus causing the phase to raise to $-90°$ [1pt]. The Bode diagram in Figure 4 can be excluded because it has the typical shape corresponding to one stable zero and one stable pole [1pt]. The Bode diagram in Figure 5 can be excluded because the second corner frequency (the one corresponding to the zero) occurs at $\omega = 100$, as can be understood from the magnitude diagram, while $P(s)$ has a zero at 2 rad/sec [1pt].

(b) Nyquist diagram

(c)

[1pt]

$$\begin{aligned} P(i\omega_{pc}) &= \frac{i0.5\omega_{pc}+1}{i\omega_{pc}(i\omega_{pc}-1)} \\ &= \frac{i0.5\omega_{pc}+1}{-\omega_{pc}^2 - i\omega_{pc}} \\ &= \frac{(i0.5\omega_{pc}+1)(-\omega_{pc}^2 + i\omega_{pc})}{\sqrt{\omega_{pc}^4+\omega_{pc}^2}} \\ &= \frac{-1.5\omega_{pc}^2 + i(\omega_{pc} - 0.5\omega_{pc}^3)}{\sqrt{\omega_{pc}^4+\omega_{pc}^2}} \end{aligned}$$

Thus the nonzero ω_{pc} is given by setting [1pt] $\omega_{pc} - 0.5\omega_{pc}^3 = 0$, that is

$$\omega_{pc} = \sqrt{2}$$

From

$$P(i\omega_{pc}) = \frac{i0.5\omega_{pc}+1}{i\omega_{pc}(i\omega_{pc}-1)}$$

we have

$$\begin{aligned} \text{[1pt] } |P(i\omega_{pc})| &= \left.\frac{\sqrt{1+\frac{\omega_{pc}^2}{4}}}{\omega_{pc}\sqrt{\omega_{pc}^2+1}}\right|_{\omega_{pc}=\sqrt{2}} \\ &= \left.\frac{1}{2}\frac{\sqrt{\omega_{pc}^2+4}}{\omega_{pc}\sqrt{\omega_{pc}^2+1}}\right|_{\omega_{pc}=\sqrt{2}} \\ &= \frac{1}{2}\frac{\sqrt{6}}{\sqrt{2}\sqrt{3}} \\ &= \frac{1}{2}. \end{aligned}$$

(d) Since the Nyquist plot is crossing the real axis at the point $(-\frac{1}{2}, 0)$, the number of net clockwise encirclements of the point $(-1, 0)$ is $N = 1$ [1 pt]. Since the number of unstable open loop poles is 1 then $P = 1$ and the number of unstable poles of the closed-loop system is $Z = 2$ [1 pt]. Thus the closed-loop system is unstable. 1 point extra if they specify the number of unstable poles.

(e)

$$[1\text{pt}]\ \frac{P}{1+P} = \frac{\frac{0.5s+1}{s(s-1)}}{1+\frac{0.5s+1}{s(s-1)}} = \frac{0.5s+1}{s^2-0.5s+1}$$

$$[1\text{pt}]\ s_{1,2} = \frac{0.5 \pm \sqrt{0.25-4}}{2},$$

which confirms that there are two unstable poles in the closed-loop system.

(f) [2pts] The denominator of the closed-loop system transfer function is

$$s^2 + (\frac{k_p}{2} - 1)s + k_p$$

The roots of the corresponding equation have strictly negative real part if and only if $k_p > 2$. Regarding the tracking properties, the pole at $s = 0$ guarantees a zero steady state error response to a step reference trajectory [1pt]. The steady state error response is given by

$$[1\text{pt}]\ e_{steady} = |\lim_{s\to 0} s \frac{1}{1+k_p\frac{0.5s+1}{s(s-1)}} \frac{1}{s^2}| = |\frac{1}{k_p\frac{1}{-1}}| = \frac{1}{|k_p|}.$$

[1pt] Thus a gain $k_p \geq \max\{2, 10\} = 10$ stabilises the system and gives a steady state error response less than 1/10.

(g) [5pts] From the Nyquist plot we see that if the gain of the P controller is strictly greater than $|P(i\omega_{pc})|^{-1} = 2$, then the number of net clockwise encirclements of the critical point becomes -1 (one counterclockwise encirclement), and therefore we have $Z = N + P = 0$, thus showing closed-loop stability.